On a Diffuse Interface Model for Non-Newtonian Two-Phase Flows with Matched Densities

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Diffuse Interface Model for two-phase flows

We consider

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(c,\mathbf{D}\mathbf{v}) + \nabla p = -\delta\operatorname{div}(\nabla c\otimes\nabla c) + f, \tag{0.1}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{0.2}$$

$$\partial_t c + \mathbf{v}\cdot\nabla c = m\Delta\mu, \tag{0.3}$$

$$\mu = \delta^{-1}\phi(c) - \delta\Delta c. \tag{0.4}$$

Here $\mathbf{v}$ is the mean velocity, $\mathbf{D}\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$, p is the pressure, and c is an order parameter related to the concentration of the fluids (e.g. the concentration difference or the concentration of one component). For simplicity we assume that $\delta = \rho = 1$.

Diffuse Interface Model for two-phase flows

We close the system by adding the boundary and initial conditions

$$\begin{aligned} \mathbf{v}|_{\partial\Omega} &= 0 && \text{on } \partial\Omega\times(0,\infty), && (0.5)\\ \partial_n c|_{\partial\Omega} = \partial_n\mu|_{\partial\Omega} &= 0 && \text{on } \partial\Omega\times(0,\infty), && (0.6)\\ (\mathbf{v},c)|_{t=0} &= (\mathbf{v}_0,c_0) && \text{in } \Omega. && (0.7) \end{aligned}$$

Basic Assumption

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^2-boundary and let $\Phi \in C([a, b]) \cap C^2((a, b))$ be such that $\phi = \Phi'$ satisfies

$$\lim_{s \to a} \phi(s) = -\infty, \qquad \lim_{s \to b} \phi(s) = \infty, \qquad \phi'(s) \geq -\alpha$$

for some $\alpha \in \mathbb{R}$. Let $m > 0$ and let $S\colon [a, b] \times \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ be such that

$$\begin{aligned} |\mathbf{S}(c, M)| &\leq C(|\operatorname{sym}(M)|^{q-1} + 1) \\ |\mathbf{S}(c_1, M) - \mathbf{S}(c_2, M)| &\leq C|c_1 - c_2|(|\operatorname{sym}(M)|^{q-1} + 1) \\ \mathbf{S}(c, M) : M &\geq \kappa|\operatorname{sym}(M)|^q - C_1 \end{aligned}$$

for all $M \in \mathbb{R}^{d\times d}$, $c, c_1, c_2 \in [a, b]$, and some $C, C_1, \kappa > 0$, $q \in (\frac{6}{5}, \infty)$.Moreover, we assume that $S(c, \cdot)\colon \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is strictly monotone for every $c \in [a, b]$.

$$(\mathbf{S}(c, M_1) - \mathbf{S}(c, M_2)) : (M_1 - M_2) > 0$$

for any $M_1, M_2 \in \mathbb{R}^{d\times d}_{sym}(M_1 \neq M_2)$.

For the following we denote

$$E_{mix}(c) = \int_\Omega \frac{|\nabla c|^2}{2}\, dx + \int_\Omega \Phi(c)\, dx.$$

Let $V_p(\Omega) = W^1_{p,0}(\Omega)^d \cap L^p_\sigma(\Omega)$,
$L^2_{(0)}(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f(x)\, dx = 0\}$,
$H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega)$, and $H^{-1}_{(0)}(\Omega) := H^1_{(0)}(\Omega)'$.

$$Q_t := \Omega \times (0, t).$$
$$Q := \Omega \times (0, \infty).$$

Cahn-Hilliard equation

We recall some results on the Cahn-Hilliard equation with convection term:

$$\partial_t c + \mathbf{v} \cdot \nabla c = m\Delta\mu \qquad \text{in } \Omega \times (0,\infty), \tag{0.8}$$

$$\mu = \phi(c) - \Delta c \qquad \text{in } \Omega \times (0,\infty), \tag{0.9}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega \times (0,\infty), \tag{0.10}$$

$$c|_{t=0} = c_0 \qquad \text{in } \Omega \tag{0.11}$$

for given c_0 with $E_{mix}(c_0) < \infty$ and $\mathbf{v} \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_0(\Omega)^d)$. Here $\phi = \Phi'$ and Φ is as in Basic Assumption.

Theorem 1 (Abels and Wilke ('06)) *Let* $\mathbf{v} \in L^2(0,\infty; H_0^1(\Omega)^d) \cap L^\infty(0,\infty; L^2_\sigma(\Omega))$*. Then for every* $c_0 \in H^1_{(0)}(\Omega)$ *with* $E_{mix}(c_0) < \infty$ *there is a unique solution* $c \in BC([0,\infty); H^1_{(0)}(\Omega))$ *of (0.8)-(0.11) with* $\partial_t c \in L^2(0,\infty; H^{-1}_{(0)}(\Omega))$ *and* $\mu \in L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega))$*. This solution satisfies*

$$E_{mix}(c(t)) + \int_{Q_t} |\nabla\mu|^2 \, d(x,\tau) = E_{mix}(c_0) - \int_{Q_t} \mathbf{v} \cdot \mu \nabla c \, d(x,\tau) \tag{0.12}$$

for all $t \in [0,\infty)$ *and*

$$\begin{aligned} &\|c\|^2_{L^\infty(0,\infty;H^1)} + \|\partial_t c\|^2_{L^2(0,\infty;H^{-1}_{(0)})} + \|\nabla\mu\|^2_{L^2(Q)} \\ &\leq \; C\left(E_{mix}(c_0) + \|\mathbf{v}\|^2_{L^2(Q)}\right) \qquad (0.13) \\ &\|c\|^2_{L^2_{\mathrm{uloc}}([0,\infty);W^2_r)} + \|\phi(c)\|^2_{L^2_{\mathrm{uloc}}([0,\infty);L^r)} \\ &\leq \; C_r\left(E_{mix}(c_0) + \|\mathbf{v}\|^2_{L^2(Q)}\right) \qquad (0.14) \end{aligned}$$

where $r = 6$ *if* $d = 3$ *and* $1 < r < \infty$ *is arbitrary if* $d = 2$*.*

Moreover, for every $R > 0$ the solution

$$c \in Y := L^2_{\text{loc}}([0,\infty); W^2_r(\Omega)) \cap H^1_{\text{loc}}([0,\infty); H^{-1}_{(0)}(\Omega))$$

depends continuously on

$$(c_0, v) \in X := H^1(\Omega) \times L^2_{\text{loc}}([0,\infty); L^2_\sigma(\Omega)) \text{ with } E_{mix}(c_0) + \|v\|_{L^2(0,\infty;H^1)} \leq R$$

with respect to the weak topology on Y and the strong topology on X.

Approximate system

$$\partial_t \mathbf{v} + \operatorname{div}(\Phi_\varepsilon(|\mathbf{v}|)\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S}(c, \mathbf{D}\mathbf{v}) + \nabla p = -\Psi_\varepsilon(\operatorname{div}(\nabla c \otimes \nabla c)) \quad \text{in } \Omega \times (0, T), \tag{0.15}$$

$$\operatorname{div} \mathbf{v} = 0, \quad \text{in } \Omega \times (0, T), \tag{0.16}$$

$$\partial_t c + (\Psi_\varepsilon \mathbf{v}) \cdot \nabla c = m\Delta\mu, \quad \text{in } \Omega \times (0, T), \tag{0.17}$$

$$\mu = \phi(c) - \Delta c. \quad \text{in } \Omega \times (0, T) \tag{0.18}$$

together with (0.5)-(0.6), where $\Psi_\varepsilon w = P_\sigma(\psi_\varepsilon * w)|_\Omega$, $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$, $\varepsilon > 0$, is a usual smoothing kernel such that $\psi(-x) = \psi(x)$ for all $x \in \mathbb{R}^n$, w is extended by 0 outside of Ω, and P_σ is the Helmholtz projection.

Existence of weak solutions of the approximate system

Theorem 2 (Abels-Diening-T.) *For every* $0 < T < \infty$, $\mathbf{v}_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ *such that* $c_0(x) \in [a,b]$ *almost everywhere there is a weak solution* $(\mathbf{v}, c, \mu)$ *of* (0.15)-(0.18),(0.5)-(0.7) *such that*

$$\mathbf{v} \in W^1_{p'}([0,T]; V_p(\Omega)') \cap L^p(0,T; V_p(\Omega)),$$
$$c \in BC([0,T]; H^1(\Omega)) \cap H^1(0,T; H^{-1}_{(0)}(\Omega)) \cap L^2(0,T; W^2_r(\Omega)),$$
$$\mu \in L^2(0,T; H^1(\Omega))$$

where $r = 6$ *if* $d = 3$ *and* $1 \leq r < \infty$ *is arbitrary if* $d = 2$.

Moreover, for every $0 \le t \le T$

$$\begin{aligned}\frac{1}{2}\|\mathbf{v}(t)\|^2_{L^2(\Omega)} + E_{mix}(c(t)) + \int_0^t \int_\Omega \mathbf{S}(c, \mathbf{Dv}) : \mathbf{Dv}\, dx\, d\tau \\ + \int_0^t \int_\Omega m|\nabla\mu|^2\, dx\, d\tau = \frac{1}{2}\|\mathbf{v}_0\|^2_{L^2(\Omega)} + E_{mix}(c_0) \\ =: E_0 \end{aligned} \tag{0.19}$$

and

$$\|c\|_{L^2(0,T;W^2_r(\Omega))} + \|\phi(c)\|_{L^2(0,T;L^r(\Omega))} \le C(T, E_0) \tag{0.20}$$

for some $C(T, E_0) > 0$ *depending continuously on* T, E_0.

Definition of Weak Solutions Let $d = 2$ or $d = 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^2-boundary and $0 < T < \infty$. Assume ϕ and $\mathbf{S}$ satisfies the Basic Assumption. Let $v_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ s.t. $c_0(x) \in [a, b]$ a.e. $x \in \Omega$. Then a triplet $(\mathbf{v}, c, \mu)$ such that

$$\mathbf{v} \in C_w([0,T]; L^2_\sigma(\Omega)) \cap W^1_{p'}([0,T]; V_p(\Omega)') \cap L^p(0,T; V_p(\Omega)),$$
$$c \in BC([0,T]; H^1(\Omega)) \cap H^1(0,T; H^{-1}_{(0)}(\Omega)) \cap L^2(0,T; W^2_r(\Omega)),$$
$$\mu \in L^2(0,T; H^1(\Omega))$$

where $r = 6$ if $d = 3$ and $1 \leq r < \infty$ is arbitrary if $d = 2$, which satisfies the following is called a weak solution of (0.1) - (0.6).

For any $\varphi \in (C^\infty(Q_T))^d$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ and $\operatorname{supp}(\varphi) \subset\subset \Omega \times [0, T)$,

$$- \int_{Q_T} \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} \; d(x,t) - \int_{Q_T} \mathbf{v} \otimes \mathbf{v} : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) + \int_{Q_T} \mathbf{S}(c, \mathbf{D}\mathbf{v}) : \mathbf{D}\boldsymbol{\varphi} \; d(x,t)$$
$$= \int_{Q_T} \nabla c \otimes \nabla c : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) + \int_{\Omega} \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0) dx \qquad (0.21)$$

holds and for $\psi \in C^\infty_{(0)}([0, T) \times \overline{\Omega})$,

$$- \int_{Q_T} c \partial_t \psi \; dx \; dt - \int_{\Omega} c_0 \psi(0) + \int_{Q_T} (v \cdot \nabla c) \psi \; dx \; dt = - \int_{Q_T} \nabla \mu \cdot \nabla \psi \; d(x,t)$$
$$\mu = \phi(c) - \Delta c,$$
$$\partial_n c|_{\partial\Omega} = 0$$

holds.

Main Theorem (Abels-Diening-T.) *Let $d = 2$ or $d = 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^2-boundary and $0 < T < \infty$. Assume ϕ and $\mathbf{S}$ satisfies the Basic Assumption. Let $v_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ s.t. $c_0(x) \in [a, b]$ a.e. $x \in \Omega$. Then there exists a weak solution of (0.1) - (0.6).*

Sketch of proof

There is a unique weak solution of the approximate system (0.15) - (0.4) together with boundary conditions. We pass that solution to the limit when ϵ tends to zero, using an adaptation of the Lipshitz truncation method, which was used for the construction of weak solutions of the power-law fluid equations with low powers in Diening-Ruzicka-Wolf ('10). Then we get a weak solution of (0.1) - (0.6).

Power-Law Fluid equations

$$\rho\partial_t \mathbf{v} + \rho\mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{v}) + \nabla p = f, \qquad (0.22)$$

$$\operatorname{div}\mathbf{v} = 0, \qquad (0.23)$$

where $S\colon \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ satisfies

$$\begin{aligned} |\mathbf{S}(M)| &\leq C(|\operatorname{sym}(M)|^{q-1} + 1) \\ \mathbf{S}(M) : M &\geq \kappa|\operatorname{sym}(M)|^q - C_1 \end{aligned}$$

for all $M \in \mathbb{R}^{d\times d}$, and some $C, C_1, \kappa > 0$, $q \in [1,\infty)$.
S is strictly monotone.

We review the known results about the weak solution of (0.22) - (0.23) (with boundary condition).

- Ladyzhenskaya ('67, '68) and Lions ('69) proved the existence of a unique weak solution when $q \geq \frac{d+2}{2}$.
- In periodic boundary condition case, Necas, Malek and Ruzicka ('93) proved the existence of a weak solution when $q > \frac{3d}{d+2}$.
- In Dirichlet boundary condition case, Necas, Malek and Ruzicka ('01) proved the existence of a weak solution when $2 \leq q < 3$ when $d = 3$.
- In Dirichlet boundary condtion case, Wolf ('07) proved the existence of a weak solution when $q > \frac{2(d+1)}{d+2}$, using L^∞-test functions and the local pressure method.

- In Dirichlet boundary condition case, Diening, Ruzicka and Wolf ('10) proved the existence of a weak solution when $q > \frac{2d}{d+2}$, using Lipschitz truncation method and the local pressure method.

Remark. When $q = 2$ (i.e. Navier-Stokes equations case), the existence of weak solutions can be proven in all dimensions more easily.

Results on Navier-Stokes-Cahn-Hilliard equations

- Abels('09)

Results on Power Law Fluid equations coupled with Cahn-Hilliard equations

- Kim-Consiglieri-Rodorigues('06)
- Grasselli-Prazak('11)

Our main result treats the case with low q which was not treated in the literatures, which corresponds to the result in Diening-Ruzicka-Wolf ('10).

Lipshitz truncation lemma

Lemma *Let*
$\mathbf{u} \in L^\infty(0, T; L^2(G)) \cap L^q(0, T; W^{1,q}(G))\,(1 < q < \infty)$ *and*
$\mathbf{H} \in L^\sigma(0, T; L^\sigma(G))\,(1 < \sigma < \infty)$ *be such that*

$$-\int_{G\times(0,T)} \mathbf{u}\cdot\partial_t\varphi\; d(x,t) \;=\; \int_{G\times(0,T)} \mathbf{H} : \nabla\varphi\; d(x,\tau) \qquad (0.24)$$

for all $\varphi \in C_0^\infty(G \times (0, T))$. *We define*

$$\mathcal{O}_\Lambda := \Big\{(x,\tau) \in \mathbb{R}^{d+1} \,\big|\, \mathcal{M}^*(|\nabla\mathbf{u}|)(x,t) + \alpha\,\mathcal{M}^*(|\mathbf{H}|)(x,t) > \Lambda\Big\}\,, \Lambda > 0\,,$$

$$\mathcal{U}_1 := \Big\{(x,t) \in \mathbb{R}^{d+1} \,\big|\, \mathcal{M}^*(|\mathbf{u}|)(x,t) > 1\Big\}\;.$$

Let $\Lambda > 0$ *and the open set* $E \subset \mathbb{R}^{d+1}$ *with* $\mathcal{L}_{d+1}(E) < \infty$ *be such that*

$$\big(G \times (0, T)\big) \cap \big(\mathcal{O}_\Lambda \cup \mathcal{U}_1\big) \subset E \subset G \times (0, T)\,. \qquad (0.25)$$

Lipshitz truncation lemma

Let $K \subset G \times (0, T)$ *be a compact set. Then we have:*
(i) *The Lipschitz truncation* $\mathcal{T}_E^{\alpha}\mathbf{u}$ *belongs to* $C^{0,1}_{d_\alpha}(K)$ *with a norm depending on* $n, K, \Lambda, \alpha, \|\mathbf{u}\|_{L^1(E)}, \|\mathbf{u}\|_{L^1(\tilde{K}\times(0,T))}$, *where* $K \subset\subset \tilde{K} \subset\subset G$. *In particular, we have* $\mathcal{T}_E^{\alpha}\mathbf{u}, \nabla\mathcal{T}_E^{\alpha}\mathbf{u} \in L^\infty(K)$.
(ii) *The Lipschitz truncation* $\mathcal{T}_E^{\alpha}\mathbf{u}$ *satisfies the estimates*

$$\|\nabla\mathcal{T}_E^{\alpha}\mathbf{u}\|_{L^\infty(K)} \leq c\left(\Lambda + \alpha^{-1}\,\delta_{\alpha,K}^{-d-3}\,\|\mathbf{v}\|_{L^1(E)}\right), \tag{0.26}$$

$$\|\mathcal{T}_E^{\alpha}\mathbf{v}\|_{L^\infty(K)} \leq c\left(1 + \alpha^{-1}\,\delta_{\alpha,K}^{-d-2}\,\|\mathbf{u}\|_{L^1(E)}\right), \tag{0.27}$$

where $\delta_{\alpha,K} := d_\alpha(K, \partial(G \times (0,T)))$ *and where the constants* c *depend only on* n. *Here* $\alpha > 0$ *and*
$d_\alpha\left((x,s),(y,t)\right) := \max\left\{|x-y|, |\alpha^{-1}(s-t)|^{\frac{1}{2}}\right\}$.

Lipschitz truncation lemma (continued)

(iii) *The function* $(\partial_t \mathcal{T}_E^\alpha \mathbf{v}) \cdot (\mathcal{T}_E^\alpha \mathbf{u} - \mathbf{u})$ *belongs to* $L^1(K \cap E)$ *and we have*

$$\left\|(\partial_t \mathcal{T}_E^\alpha \mathbf{v}) \cdot (\mathcal{T}_E^\alpha \mathbf{u} - \mathbf{u})\right\|_{L^1(K\cap E)} \le c\, \alpha^{-1} \mathcal{L}_{d+1}(E) \left(\Lambda + \alpha^{-1}\, \delta_{\alpha,K}^{-d-3} \|\mathbf{u}\|_{L^1(E)}\right)^2, \tag{0.28}$$

where the constant c *depends only on* n.

(iv) *For all* $\zeta \in C_0^\infty(G \times (0,T))$ *holds the identity*

$$\int_0^T \left\langle \partial_t \mathbf{u}(t), (\mathcal{T}_E^\alpha \mathbf{u}(t))\, \zeta(t) \right\rangle dt \tag{0.29}$$

$$= \frac{1}{2} \int_{G\times(0,T)} \left(|\mathcal{T}_E^\alpha \mathbf{u}|^2 - 2\mathbf{u} \cdot \mathcal{T}_E^\alpha \mathbf{u} \right) \partial_t \zeta \, d(x,t) \tag{0.30}$$

$$+ \int_E (\partial_t \mathcal{T}_E^\alpha \mathbf{u}) \cdot (\mathcal{T}_E^\alpha \mathbf{u} - \mathbf{u}) \zeta \, d(x,t),$$

where $\langle \cdot, \cdot \rangle$ *denotes the usual duality pairing with respect to* G.

The existence of weak solutions of the approximate system follows from Theorem 2. Using the a priori estimates given by (0.19) and (0.20), we can conclude for a suitable subsequence $\varepsilon_i \to_{i\to\infty} 0$ that

$$\begin{aligned} \mathbf{Dv}_{\varepsilon_i} &\to \mathbf{Dv} \;\; \text{weakly in} \;\; L^q(Q_T), \\ \mathbf{v}_{\varepsilon_i} &\to \mathbf{v} \;\; \text{weakly in} \;\; L^{q\frac{d+2}{d}}(Q_T), \\ \mathbf{S}\left(c_{\varepsilon_i}, \mathbf{Dv}_{\varepsilon_i}\right) &\to \widetilde{\mathbf{S}} \;\; \text{weakly in} \;\; L^{q'}(Q_T), \\ \mathbf{v}_{\varepsilon_i} \otimes \mathbf{v}_{\varepsilon_i} \Phi_{\varepsilon_i}\left(\mathbf{v}_{\varepsilon_i}\right) &\to \widetilde{\mathbf{H}} \;\; \text{weakly in} \;\; L^{q\frac{d+2}{2d}}(Q_T). \end{aligned} \tag{0.31}$$

Moreover, because of (0.20), (0.17), and the Lemma of Aubin-Lions, it is easy to prove that

$$\nabla c_{\varepsilon_i} \to_{i\to\infty} \nabla c \qquad \text{in } L^2(0, T; C^1(\overline{\Omega}))$$

since $W_6^2(\Omega) \hookrightarrow C^1(\overline{\Omega})$ compactly. Interpolation with the boundedness of $c_\varepsilon \in L^\infty(0, T; H^1(\Omega))$ yields

$$\nabla c_{\varepsilon_i} \to_{i\to\infty} \nabla c \qquad \text{in } L^4(Q_T). \tag{0.32}$$

Let $\mathbf{K}_\varepsilon \in L^2(Q_T)^{d\times d}$ be such that

$$\int_{Q_T} \mathbf{K}_\varepsilon : \mathbf{D}\varphi \, d(x,\tau) = \int_{Q_T} (\nabla c_\varepsilon \otimes \nabla c_\varepsilon) : \mathbf{D}\Psi_\varepsilon(\varphi) \; d(x,\tau) \quad (0.33)$$
$$= -\int_{Q_T} \Psi_\varepsilon \left(\operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon)\right) \cdot \boldsymbol{\varphi} \; d(x,\tau)$$

for all $\boldsymbol{\varphi} \in L^2(0,T; H_0^1(\Omega)^d)$ and that $\mathbf{K}_\varepsilon \in L^2(Q_T)^d$ depends continuously on $\Psi_\varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \in L^2(0,T; H_0^{-1}(\Omega)^d)$. Then

$$\mathbf{K}_{\varepsilon_i} \to \mathbf{K} := \nabla c \otimes \nabla c \;\text{ strongly in }\; L^2(Q_T)^{d\times d},$$

due to (0.32).

We consider only the case $q < 2$ for simplicity. Next, let $G \subset\subset \Omega$ be a fixed but arbitrary open bounded set. Clearly we may assume there exists an open bounded set $G' \subset\subset \Omega$ with $G \subset\subset G'$ and $\partial G' \in C^2$. Similarly as in Diening-Ruzicka-Wolf, we have for some $\varepsilon_i \to_{i\to\infty} 0$,

$$\mathbf{v}_{\varepsilon_i} \to \mathbf{v} \qquad \text{strongly in} \;\; L^{2\sigma_0}(0, T; L^{2\sigma_0}(G')) \tag{0.34}$$

$$\text{and} \;\; \mathbf{v}_{\varepsilon_i} \otimes \mathbf{v}_{\varepsilon_i} \Phi_{\varepsilon_i}(|\mathbf{v}_\varepsilon|) \to \mathbf{v} \otimes \mathbf{v} \qquad \text{strongly in} \;\; L^{\sigma_0}(0, T; L^{\sigma_0}(G')), \tag{0.35}$$

where $\sigma_0 > 1$ and $q \leq 2\sigma_0 < q\frac{d+2}{d}$. We also have for $i \to \infty$,

$$\mathbf{v}_{\varepsilon_i} \to \mathbf{v} \quad \text{strongly in} \;\; L^r(0, T; L^2(G')), \text{ for all } 1 \leq r < \infty \tag{0.36}$$

by interpolation of (0.34) with the boundedness of $(\mathbf{v}_\varepsilon) \in L^\infty(0, T; L^2(\Omega))$.

Taking the limit of the weak form of the approximate system along the subsequence ε_i, we have the following.

$$
\begin{aligned}
-\int_{Q_T} \mathbf{v} \cdot \partial_\tau \boldsymbol{\varphi} \; d(x,t) + \int_{Q_T} (\widetilde{\mathbf{S}} - \mathbf{v} \otimes \mathbf{v}) : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) \\
= \int_{Q_T} \mathbf{K} : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) + \int_{\Omega} \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0) \; dx.
\end{aligned} \tag{0.37}
$$

By subtracting the above equation from the weak form of the approximate equations, we have the following.

$$
\begin{aligned}
-\int_{Q_T} (\mathbf{v}_\varepsilon - \mathbf{v}) \; \cdot \; \partial_t \boldsymbol{\varphi} \; d(x,t) + \int_{Q_T} \left(\mathbf{S}(c_\varepsilon, \mathbf{D}u_\varepsilon) - \widetilde{\mathbf{S}} \right) : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) \\
= \int_{Q_T} (\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \Phi_\varepsilon(\mathbf{v}_\varepsilon) - \mathbf{v} \otimes \mathbf{v}) : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) \\
+ \int_{Q_T} (\mathbf{K}_\varepsilon - \mathbf{K}) : \mathbf{D}\boldsymbol{\varphi} \, d(x,t).
\end{aligned} \tag{0.38}
$$

Using a local pressure decomposition method as in Diening-Ruzicka-Wolf, one gets unique functions:

$$\begin{aligned} p_{1,\varepsilon} &\in L^{q'}\left((0,T);L^{q'}(G')\right), \\ p_{2,\varepsilon} &\in L^{\sigma_0}\left((0,T);L^{\sigma_0}(G')\right), \\ p_{3,\varepsilon} &\in L^{2}\left((0,T);L^{2}(G')\right) \quad \text{and} \\ p_{h,\varepsilon} &\in C_w\left([0,T];W^{1,2}(G')\right) \end{aligned} \tag{0.39}$$

with $\Delta p_{h,\varepsilon} = 0$, and $p_{h,\varepsilon}(0) = 0$ and

$$
\begin{aligned}
&-\int_0^T \int_{G'} (\mathbf{v}_\varepsilon - \mathbf{v}) \cdot \partial_t \boldsymbol{\varphi} \, dx \, dt + \int_0^T \int_{G'} \left(\mathbf{S}(c_\varepsilon, \mathbf{D} v_\varepsilon) - \widetilde{\mathbf{S}} \right) : \nabla \boldsymbol{\varphi} \, dx \, dt \\
&\quad = \int_0^T \int_{G'} \left(\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \Phi_\varepsilon(|\mathbf{v}_\varepsilon|) - \mathbf{v} \otimes \mathbf{v} \right) : \nabla \boldsymbol{\varphi} \, dx \, dt \\
&\quad + \int_0^T \int_{G'} (\mathbf{K}_\varepsilon - \mathbf{K}) : \nabla \boldsymbol{\varphi} \, dx \, dt \\
&\quad + \int_0^T \int_{G'} \left\{ (p_{1,\varepsilon} + p_{2,\varepsilon} + p_{3,\varepsilon}) \operatorname{div} \boldsymbol{\varphi} + \nabla p_{h,m} \cdot \partial_t \boldsymbol{\varphi} \right\} \, dx \, dt
\end{aligned}
\tag{0.40}
$$

for all $\boldsymbol{\varphi} \in \left(C_0^\infty(G' \times (0, T)) \right)^d$.

$$\|p_{1,\varepsilon}\|_{L^{q'}(G'\times(0,T))} \le C\|\mathbf{S}(c_\varepsilon, \mathbf{D}\mathbf{v}_\varepsilon) - \widetilde{\mathbf{S}}\|_{L^{q'}(G'\times(0,T))}, \tag{0.41}$$

$$\|p_{2,\varepsilon}\|_{L^{\sigma_0}(G'\times(0,T))} \le C\|\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \Phi_\varepsilon(|\mathbf{v}_\varepsilon|) - \mathbf{v}\otimes\mathbf{v}\|_{L^{\sigma_0}(G'\times(0,T))}, \tag{0.42}$$

$$\|p_{3,\varepsilon}\|_{L^2(G'\times(0,T))} \le C\|\mathbf{K}_\varepsilon - \mathbf{K}\|_{L^2(Q_T)} \text{ and} \tag{0.43}$$

$$\|p_{h,\varepsilon}(t)\|_{W^{1,2}(G')} \le C\|\mathbf{v}_\varepsilon(t) - \mathbf{v}(t)\|_{L^2(G')}, \qquad t \in (0, T). \tag{0.44}$$

Since $p_{h,\varepsilon}$ is harmonic in G', as in Diening-Ruzicka-Wolf, it follows that for all $t \in (0, T)$ and all $1 \leq r \leq \infty$,

$$\begin{aligned} \|p_{h,\varepsilon}(t)\|_{W^{2,r}(G)} &\leq C\|p_{h,\varepsilon}(t)\|_{L^2(G')} \\ &\leq C\|\mathbf{v}_\varepsilon(t) - \mathbf{v}(t)\|_{L^2(G')} \end{aligned} \tag{0.45}$$

where the constant depends on d, G' and G.
If we set $\mathbf{u}_\varepsilon := (\mathbf{v}_\varepsilon - \mathbf{v} + \nabla p_{h,\varepsilon})\chi_{G\times(0,T)}$, we have

$$\mathbf{u}_\varepsilon \to 0 \quad \text{strongly in } L^{2\sigma_0}(G \times (0, T)) \;\; \varepsilon \to 0.$$

We can also see that (0.40) can be rewritten for any $\boldsymbol{\varphi} \in (C_0^\infty(G \times (0, T))^d$ as follows.

$$
\begin{aligned}
&- \int_0^T \int_G \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} \; d(x,t) + \int_0^T \int_G \Big(\mathbf{S}(c_\varepsilon, \mathbf{D}\mathbf{v}_\varepsilon) - \widetilde{\mathbf{S}} \Big) : \mathbf{D}\boldsymbol{\varphi} \; d(x,t) \\
&= \int_0^T \int_G (\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \Psi_\varepsilon(|\mathbf{v}_\varepsilon|) - \mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\varphi} \; dx \; dt \\
&+ \int_{Q_T} (\mathbf{K}_\varepsilon - \mathbf{K}) : \mathbf{D}\boldsymbol{\varphi} \, d(x,t) \\
&+ \int_0^T \int_G (p_{1,\varepsilon} + p_{2,\varepsilon} + p_{3,\varepsilon}) \operatorname{div} \boldsymbol{\varphi} \; d(x,t) \qquad (0.46)
\end{aligned}
$$

From this, we get $\partial_t \mathbf{u}_\varepsilon \in L^{\sigma_0}(0, T; W^{-1,\sigma_0}(G)))$. Therefore, if we put

$$\begin{aligned}
\mathbf{H}_{1,\varepsilon} &= \widetilde{\mathbf{S}} - \mathbf{S}(c_\varepsilon, \mathbf{D}\mathbf{u}_\varepsilon) + p_{1,\varepsilon}\mathbf{I}, \\
\mathbf{H}_{2,\varepsilon} &= \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon \Psi_\varepsilon(|\mathbf{v}_\varepsilon|) - \mathbf{v} \otimes \mathbf{v} + p_{2,\varepsilon}\mathbf{I}, \\
\mathbf{H}_{3,\varepsilon} &= \mathbf{K}_\varepsilon - \mathbf{K} + p_{3,\varepsilon}\mathbf{I}, \\
\text{and} \quad \mathbf{H}_\varepsilon &= \mathbf{H}_{1,\varepsilon} + \mathbf{H}_{2,\varepsilon} + \mathbf{H}_{3,\varepsilon},
\end{aligned}$$

then (0.46) can be written both as

$$-\int_{G\times(0,T)} \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} \; d(x,t) = \int_{G\times(0,T)} \mathbf{H}_\varepsilon : \nabla \boldsymbol{\varphi} \; d(x,t) \quad (0.47)$$

for all $\boldsymbol{\varphi} \in (C_0^\infty(G \times (0, T)))^d$ and as

$$\int_0^T \langle \partial_t \mathbf{u}_\varepsilon, \boldsymbol{\varphi} \rangle \; dt = \int_{G\times(0,T)} \mathbf{H}_\varepsilon : \nabla \boldsymbol{\varphi} \; d(x,t) \quad (0.48)$$

for all $\boldsymbol{\varphi} \in \left(L^{\sigma_0'} \left(0, T; W_0^{1,\sigma_0'}(G) \right) \right)^d$.

We define the set $E_{k,\epsilon}$ and $\alpha_{k,\epsilon}$ where $k \in \mathbb{N}$ appropriately by using $\mathbf{u}_\epsilon$ and $\mathbf{H}_{1,\varepsilon}$, $\mathbf{H}_{2,\varepsilon}$, $\mathbf{H}_{3,\varepsilon}$. Then we can use Lipschitz truncation lemma by setting $\mathbf{u} = \mathbf{u}_\varepsilon$, $\mathbf{H} = \mathbf{H}_\varepsilon$, $E = E_{k,\varepsilon}$ and $\alpha = \alpha_{k,\varepsilon}$.

We choose $k \in \mathbb{N}$ appropriately for each ε and setting it to be ε_k. Letting $k \to \infty$ in the equality which is obtained by Lipschitz truncation lemma we get

$$\lim_{k\to\infty} \int_{G\times(0,T)} \mathbf{S}(c, \mathbf{Dv}_{\epsilon_k}) : \mathbf{Dv}_{\epsilon_k} \zeta d(x,t) = \int_{G\times(0,T)} \widetilde{\mathbf{S}} : \mathbf{Dv} \zeta d(x,t). \tag{0.49}$$

With the help of the local Minty trick we obtain

$$\widetilde{\mathbf{S}}\zeta = \mathbf{S}(c, \mathbf{Dv})\zeta \quad \text{a.e. in} \quad G \times (0, T). \tag{0.50}$$

Hence

$$\widetilde{\mathbf{S}} = \mathbf{S}(c, \mathbf{Dv}) \quad \text{a.e. in} \quad G \times (0, T).$$